Shape Optimization of Rotating Cantilever Beams Considering Their Varied Modal Characteristics

Jung Eun Cho, Hong Hee Yoo*
School of Mechanical Engineering, Hanyang University, Haengdang-Dong 17, Sungdong-Gu, Seoul 133-791, Korea

The modal characteristics of rotating structures vary with the rotating speed. The material and the geometric properties of the structures as well as the rotating speed influence the variations of their modal characteristics. Very often, the modal characteristics of rotating structures need to be specified at some rotating speeds to meet their design requirements. In this paper, rotating cantilever beam is chosen as a design target structure. Optimization problems are formulated and solved to find the optimal shapes of rotating beams with rectangular cross section.

Key Words : Shape Optimization, Rotating Cantilever Beam, Modal Characteristics

1. Introduction

When cantilever beam-like structures such as turbine and helicopter blades are designed, their natural frequencies need to be estimated to avoid undesirable problems such as resonance. It is a common practice to find the natural frequencies of a non-rotating structure if the geometric shape and the material properties of the structure are given. However, such a structure normally rotates during its operation, and the rotational motion significantly influences the modal characteristics of the structure. So, if the modal characteristics of stationary structures are indiscriminately used for their design, critical design failures may occur due to the modal characteristics variations which are resulted from rotational motion. Therefore, the varied modal characteristics of rotating structures need to be considered for their safe and reliable designs.

The modal characteristics of rotating cantilever beams were first investigated by Southwell and Gough (1921), and their monumental work was followed by many theoretical and numerical studies (Yntema, 1955 ; Schilhansl, 1958 ; Carnegie, 1959 ; Putter and Manor, 1978). Recently, more elaborate methods (Kane et al., 1987 ; Yoo et al., 1995 ; Yoo and Shin, 1998) have been introduced to analyze the modal characteristics of rotating cantilever beams efficiently and in detail. Using the methods proposed so far, the modal characteristics of rotating cantilever beams could be analyzed if the geometric and the material properties are given. For a practical design of a rotating structure, some specific modal characteristics are given as design requirements (to avoid undesirable excessive vibration problems), and the geometric shape of the structure needs to be found. Such investigations for rotating structures are, however, rarely found in the literature.

The purpose of this paper is to find the optimal cross section shapes of rotating cantilever beams that provide some specific modal characteristics such as maximal and minimal natural frequencies versus the angular speed. Only the thickness shapes of rotating beams were investigated in the previous study (Yun and Yoo, 2001). Chordwise bending equations of motion of rotating beams were employed for the investigation. To extend the previous study, both the thickness and the

* Corresponding Author,
E-mail : hhyoo@hanyang.ac.kr
TEL : +82-2-2290-0446; **FAX** : +82-2-2293-5070
School of Mechanical Engineering, Hanyang University, Haengdang-Dong 17, Sungdong-Gu, Seoul 133-791, Korea. (Manuscript **Received** June 12, 2003; **Revised** November 17, 2003)

width shapes are considered in the present study by using flapwise bending equations of rotating beams. To derive the equations of motion, Kane's method (Kane and Levinson, 1985) along with the assumed mode method is employed. MFD (Method of Feasible Directions) as an optimization algorithm (Arora, 1989; Vanderplaates Research and Development, 1995) and CDM (Central Difference Method) for the sensitivity analysis are also utilized to solve the optimization problems.

2. Equations of Motion

In this section, equations of motion of rotating cantilever beams are derived by using the linear dynamic modeling that employs hybrid deformation variables proposed in Yoo et al. (1995). The following assumptions are employed in the present study. First, the beam has homogeneous and isotropic material properties. Second, the beam has slender shape with rectangular cross section so that shear and rotary inertia effects can be neglected. These assumptions are employed to avoid complexities involved in more general geometric shapes and material properties of beam and to achieve the purpose of the present study, which is how to formulate design problems for rotating beams and to find the optimal solutions of the problems.

Figure 1 shows the configuration of a cantilever beam fixed in a rigid hub (reference frame A) which rotates with angular speed. In the figure, $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ represent orthogonal unit vectors fixed in the rigid hub, $\vec{u}$ is the elastic deformation vector of a generic point, and u_1, u_2, and u_3 are the components of the elastic deformation vector.

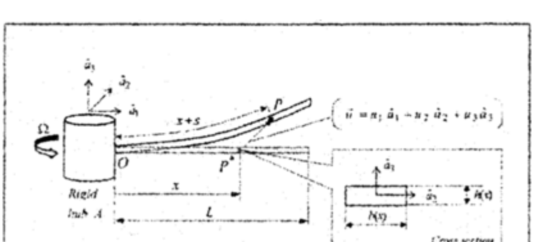

Fig. 1 Configuration of a rotating cantilever beam

The angular velocity of the rigid hub and the velocity of a point O (which is fixed in A) can be expressed as follows:

$$\vec{\omega}^A = \Omega \hat{a}_3,\ \vec{v}^O = r\Omega \hat{a}_2 \tag{1}$$

Thus, the velocity of the generic point P can be expressed as follows:

$$\begin{aligned}\vec{v}^P = & [\dot{u}_1 - \Omega u_2]\hat{a}_1 \\ & + [\dot{u}_2 + \Omega(r + x + u_1)]\hat{a}_2 + \dot{u}_3\hat{a}_3\end{aligned} \tag{2}$$

where r is the radius of the rigid hub and x is the distance from point O to the generic point in undeformed configuration. In the present work, s, u_2 and u_3 are approximated by employing the assumed mode method as follows:

$$\begin{aligned} s(x,\ t) &= \sum_{j=1}^{\mu_1} \phi_{1j}(x)\, q_{1j}(t) \\ u_2(x,\ t) &= \sum_{j=1}^{\mu_2} \phi_{2j}(x)\, q_{2j}(t) \\ u_3(x,\ t) &= \sum_{j=1}^{\mu_3} \phi_{3j}(x)\, q_{3j}(t) \end{aligned} \tag{3}$$

where ϕ_{1j}, ϕ_{2j} and ϕ_{3j} are spatial functions, q_{1j}, q_{2j} and q_{3j} are generalized coordinates, and μ_1, μ_2 and μ_3 are the numbers of the generalized coordinates. Since the non-Cartesian variable s denoting the arc-length stretch is approximated, the following geometric relation is used to derive the equations of motion

$$x + s = \int_0^x \left[\left(1 + \frac{\partial u_1}{\partial \sigma}\right)^2 + \left(\frac{\partial u_2}{\partial \sigma}\right)^2 + \left(\frac{\partial u_3}{\partial \sigma}\right)^2\right]^{1/2} d\sigma \tag{4}$$

Now, the equations of motion can be derived from the following equation

$$\int_0^L \rho b h \left(\frac{\partial \vec{v}^P}{\partial \dot{q}_i}\right) \cdot \frac{d\vec{v}^P}{dt}\, dx + \frac{\partial U}{\partial q_i} = 0 \qquad (i = 1,\ 2,\ \cdots,\ \mu) \tag{5}$$

where ρ and L are the density and the length, respectively. And q_i, consists of q_{1j}, q_{2j} and q_{3j}, and μ is the total sum of μ_1, μ_2 and μ_3. The cross section shape of the beam is rectangular and it is assumed that the thickness h and the width b of the beam are represented by cubic spline functions such that

$$\begin{aligned} h(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ b(x) &= d_0 + d_1 x + d_2 x^2 + d_3 x^3 \end{aligned} \tag{6}$$

The strain energy of the beam can be expressed as

$$U=\frac{1}{2}\int_0^L EA\left(\frac{\partial s}{\partial x}\right)^2 dx +\frac{1}{2}\int_0^L EI_{zz}\left(\frac{\partial^2 u_2}{\partial x^2}\right)^2 dx+\frac{1}{2}\int_0^L EI_{yy}\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2 dx \tag{7}$$

where E denotes the Young's modulus, A is the cross sectional area, and I_{yy} and I_{zz} are the second area moments of the cross section.

In this study, the modal characteristics of rotating beams in flapwise bending motion are of major concern. If the point O in Fig. 1 is the center of rotation, governing equations for flapwise bending motion can be derived as follows:

$$\sum_{j=1}^{\mu_3}\left(\int_0^L \rho bh\phi_{3i}\phi_{3j}dx\right)\ddot{q}_{3j}+\sum_{j=1}^{\mu_3}\left(\int_0^L E\frac{bh^3}{12}\phi_{3i,xx}\phi_{3j,xx}dx\right)q_{3j} +\sum_{j=1}^{\mu_3}\Omega^2\left(\int_0^L \rho bhx\int_0^x \phi_{3i,\sigma}\phi_{3j,\sigma}d\sigma\, dx\right)q_{3j}=0 \quad (i=1, 2, \cdots, \mu_3) \tag{8}$$

By substituting Eq. (6) into Eq. (8) and using the rule of integration by parts, equations for flapwise bending motion can be written (by using matrix notations) as

$$[M^{33}]\{\ddot{q}_3\}+\langle[K^B]+\Omega^2[K^G]\rangle\{q_3\}=0 \tag{9}$$

where $[M^{33}]$, $[K^B]$, and $[K^G]$ are defined by

$$M_{ij}^{33}\equiv\int_0^L \rho(a_0+a_1x+a_2x^3+a_3x^3)(d_0+d_1x+d_2x^2+d_3x^3)\,\phi_{3i}\phi_{3j}dx \tag{10}$$

$$K_{ij}^B\equiv\int_0^L \frac{E}{12}(d_0+d_1x+d_2x^2+d_3x^3)(a_0+a_1x+a_2x^3+a_3x^3)^3\phi_{3i,xx}\phi_{3j,xx}dx \tag{11}$$

$$\begin{aligned}K_{ij}^G\equiv\int_0^L \rho\Big\{&\frac{a_0d_0}{2}(L^2-x^2)+\frac{a_0d_1+a_1d_0}{3}(L^3-x^3)\\&+\frac{a_0d_2+a_1d_1+a_2d_0}{4}(L^4-x^4)\\&+\frac{a_0d_3+a_1d_2+a_2d_1+a_3d_0}{5}(L^5-x^5)\\&+\frac{a_1d_3+a_2d_2+a_3d_1}{6}(L^6-x^6)\\&+\frac{a_2d_3+a_3d_2}{7}(L^7-x^7)\\&+\frac{a_3d_3}{8}(L^8-x^8)\Big\}\phi_{3i,x}\phi_{3j,x}dx\end{aligned} \tag{12}$$

3. Optimization Problems and Numerical Results

The natural frequencies of rotating beams are determined by their angular speed and the coefficients of the cubic spline functions of thickness and width of the beam cross section. Therefore, they can be expressed as

$$\omega_k=\omega_k(\Omega, X) \tag{13}$$

where X consists of the cubic spline coefficients a_0, a_1, a_2, a_3, d_0, d_1, d_2 and d_3 which are design variables. The material and geometric properties of the beam which influence the modal characteristics are given in Table 1.

Table 1 Material and geometric properties of the beam

Young's modulus (E)	69.0 GPa
Density (ρ)	2.71×10^3 kg/m^3
Length (L)	0.4 m

The fundamental natural frequency of a constantly rotating beam is a function of the cubic spline coefficients determining the cross section shape of the beam. To find the cross section shape of the beam, which provides the minimum (or maximum) fundamental natural frequency, the objective function and the constraints are given by

$$\begin{aligned}&Min\ (or\ Max)\ \omega_1(\Omega_s, X)\\&subject\ to\ \int_0^L h(X, x)\,b(X, x)\,dx\le Lh_0b_0\\&\qquad h(X, x)\ge h_{\min}\ (0\le x\le L)\\&\qquad b(X, x)\ge b_{\min}\ (0\le x\le L)\end{aligned} \tag{14}$$

where Ω_s is the magnitude of steady state angular speed, h_0 ($=0.002$ m) and b_0 ($=0.035$ m) are reference thickness and width, and $h_{\min}$ ($=0.001$ m) and $b_{\min}$ ($=0.0175$ m) are the minimum thickness and width of the beam.

Figure 2 shows the minimum and the maximum fundamental natural frequency loci of the rotating beam when the steady-state angular speed increases from 0 rad/s to 300 rad/s. Thus, the two loci embrace the possible region of the fundamental natural frequency. To obtain Fig. 2, the optimization problem to find the minimum or the maximum fundamental natural frequency, when the angular speed is 0, is first solved with the initial values of design variables given in Table 2.

Table 2 Initial and optimum values of the design variables and the objective function (to find the minimum frequency shape when the angular speed is 0)

	Values \ Coefficients	a_0 / d_0	a_1 / d_1	a_2 / d_2	a_3 / d_3
Design variables	Initial values	0.002	0	0	0
		0.035	0	0	0
	Optimum values	0.00105	−0.00128	−0.00041	0.14720
		0.0175	0.01488	0.00652	0.00246
Objective function	Initial value	64.019			
	Optimum value	13.198			

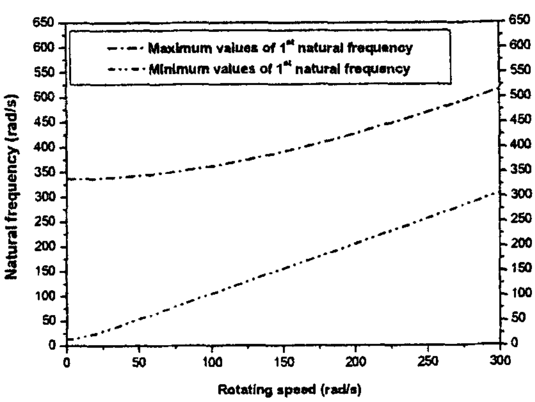

Fig. 2 Band of the fundamental natural frequency versus rotating speed

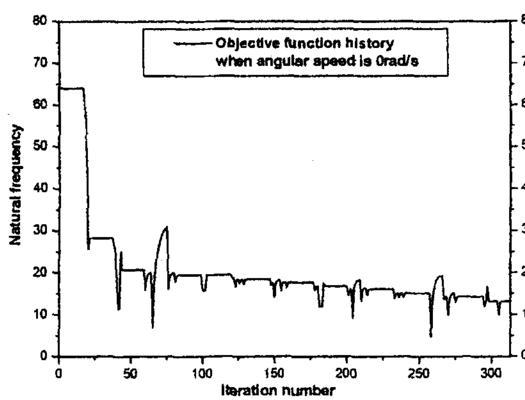

Fig. 3 Objective function history versus iteration number

The optimum values of the design variables recorded in Table 2 are then employed as the initial values to solve the optimization problem to find the minimum or the maximum fundamental natural frequency, when the angular speed is 0.1 rad/s. The same process continues until the angular speed reaches 300 rad/s. Thus, 3001 optimization problems are solved to obtain the minimum and the maximum loci, respectively. Figure 3 shows a typical objective function history while solving an optimization problem (to find the minimum frequency shape when the angular speed is 0 rad/s). Figures 4 and 5 show the thickness and the width of the beam versus the beam length for the minimum and the maximum fundamental natural frequency results. The thickness and the width variations along the beam length with angular speed 300 rad/s are shown in the figures. Both the thickness and the width vary for the minimum frequency result.

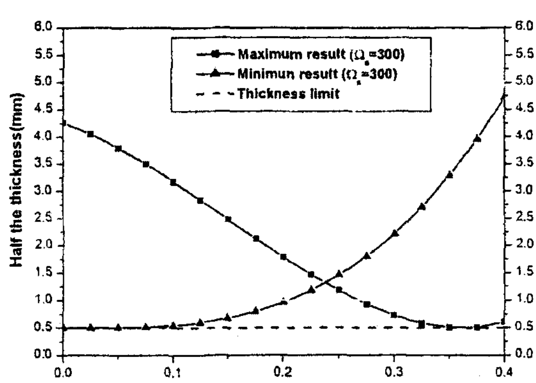

Fig. 4 Half the thickness shapes which maximize and minimize the fundamental natural frequency

However, for the maximum frequency result, only the thickness varies and the width becomes the minimum value b_{min}. Thus, it can be concluded that increasing the thickness while reducing the

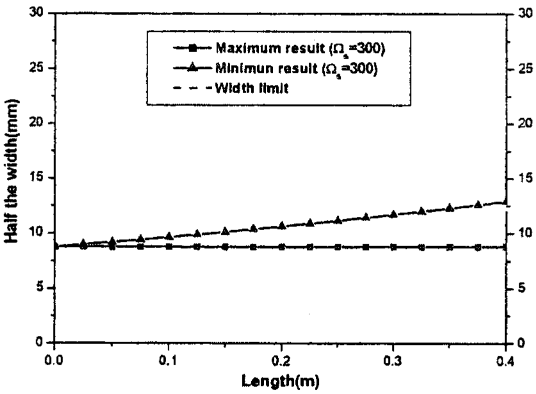

Fig. 5 Half the width shapes which maximize and minimize the fundamental natural frequency

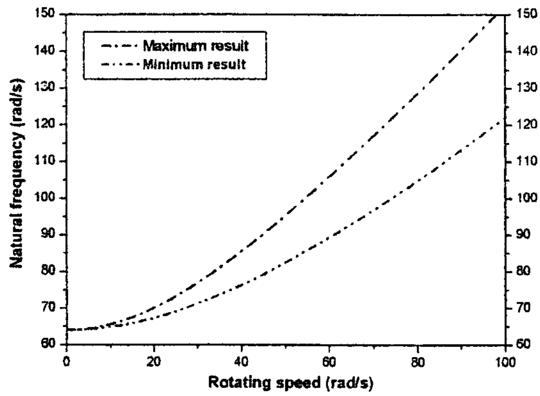

Fig. 6 Band of the fundamental natural frequency versus rotating speed

width to the minimum value is the most effective way to maximize the fundamental natural frequency.

In the following problem, another constraint is added to the previous problem. When the beam does not rotate, the fundamental natural frequency should remain constant regardless of its shape variation. So, the objective function and constraints are given as follows :

$$
\begin{aligned}
&\textit{Min (or Max)}\ \omega_1(\Omega_s, X) \\
&\textit{subject to}\ \omega_1(0, X) - \omega_1(0, X_0) = 0 \\
&\qquad \int_0^L h(X, x)\, b(X, x)\, dx \leq L h_0 b_0 \qquad (15) \\
&\qquad h(X, x) \geq h_{\min}\ (0 \leq x \leq L) \\
&\qquad b(X, x) \geq b_{\min}\ (0 \leq x \leq L)
\end{aligned}
$$

where $X_0 = [0.002, 0, 0, 0, 0.035, 0, 0, 0]$ represents the design variables of the beam with uniform cross section along its length.

Figure 6 shows the minimum and the maximum fundametal natural frequency loci of the rotating beam when the angular speed increases from 0 rad/s to 100 rad/s. Thus, the two loci embrace the possible region of the fundametal natural frequency. As shown in the figure, the two loci meet when the angular speed is zero. Figures 7 and 8 show the thickness and the width of the beam versus the length of the beam for the minimum and the maximum fundametal natural frequency results. The thickness and the width variations along the beam length with angular speed 100 rad/s are shown in the figures. Differing from

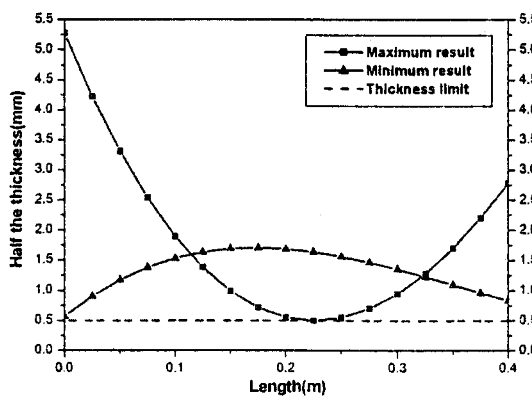

Fig. 7 Half the thickness shapes which maximize and minimize the fundamental natural frequency

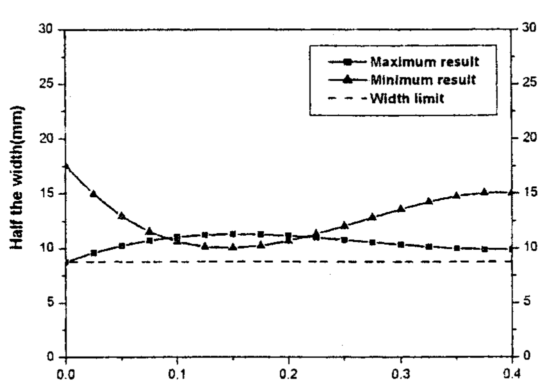

Fig. 8 Half the width shapes which maximize and minimize fundamental natural frequency

the previous results, both the thickness and the

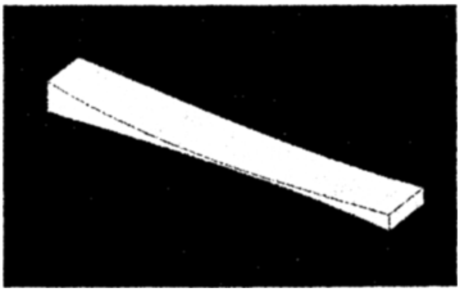

Fig. 9 Beam shape which maximizes fundamental natural frequency

Fig. 10 Beam shape which maximizes fundamental natural frequency

width vary for the minimum and the maximum frequency results. Figures 9 and 10 show the overall pictures of the beams for the minimum and the maximum frequency results, respectively. The cross sections are amplified three times compared to the lengths of the beams.

In the following, a shape of beam which has a specific natural frequency at a specific angular speed is to be found. To find such an optimal shape, the objective function is given by

$$Min[\omega_1(\Omega_G, X) - \omega_G]^2 \tag{16}$$

where Ω_G and ω_G are the specified angular speed and first natural frequency. The constraints employed for this optimization problem are same as those of Eq. (15).

Figure 11 shows three loci: the maximum fundametal natural frequency locus, the minimum fundametal natural frequency locus, and the locus which satisfies $\omega_1(\Omega_G, X) - \omega_G = 0$. For this problem, Ω_G is set by 80 rad/s and ω_G is given as 120 rad/s. Figures 12 and 13 show the thickness and the width of the beam versus the length of the beam for the specified first natural frequency result.

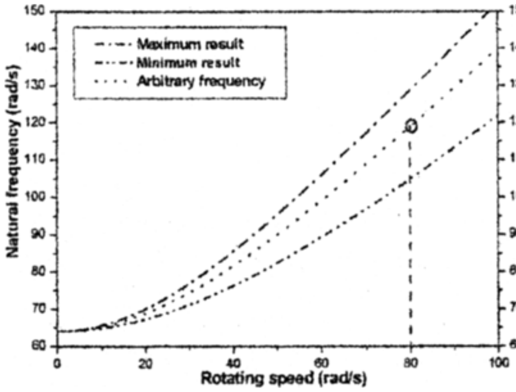

Fig. 11 Specified fundamental natural frequency locus along with the maximum and the minimum loci

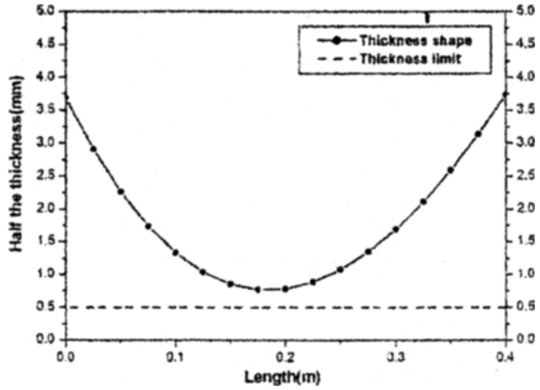

Fig. 12 Half the thickness shape of the beam which satisfies the specified first natural frequency

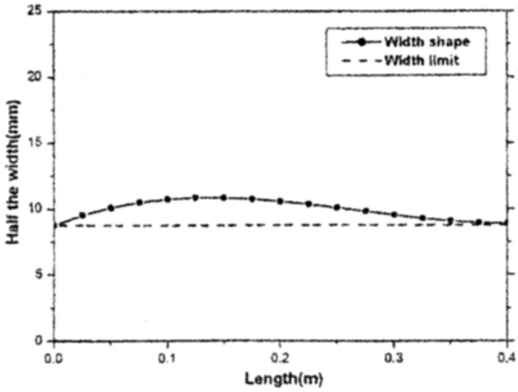

Fig. 13 Half the width shape of the beam which satisfies the specified first natural frequency

4. Conclusion

In this study, an optimization method has been employed to find rectangular cross section shapes of rotating cantilever beams which have some specific modal characteristics. The thickness and the width were represented by cubic spline functions and the coefficients of the spline functions were employed as design variables. Optimization problems were formulated and solved to find the design variables. The numerical results showed that there exist specific cross section shapes which minimize or maximize the natural frequency of the rotating beams. There also existed a specific cross section shape which has a specific natural frequency at a specific angular speed. As long as the natural frequency is specified within the band between the maximum and the minimum natural frequencies, the corresponding cross section shape can be found through the optimization procedure proposed in this work.

Acknowledgment

This research was supported by the Innovative Design Optimization Technology Engineering Research Center through the research fund, for which the authors are grateful.

References

Arora, J., 1989, *Introduction To Optimum Design*, McGraw-Hill Book Co., New York, N.Y.

Carnegie, W., 1959, "Vibrations of Rotating Cantilever Blading : Theoretical Approaches to the Frequency Problem Based on Energy Methods," *J. Mechanical Engineering Sci.*, Vol. 1, pp. 235~240.

Kane, T. and Levinson, D., 1985, *Dynamics : Theory and Applications*, McGraw-Hill Book Co., New York, N.Y.

Kane, T., Ryan, R. and Banerjee, A., 1987, "Dynamics of Cantilever Beam Attached to a Moving Base," *J. of Guidance, Control, and Dynamics*, Vol. 10, pp. 139~151.

Putter, S. and Manor, H., 1978, "Natural Frequencies of Radial Rotating Beams," *J. Sound and Vibration*, Vol. 56, pp. 175~185.

Schilhansl, M., 1958, "Bending Frequency of a Rotating Cantilever Beam," J. of Appl. Mech. Trans. Am. Soc. Mech. Engrs, Vol. 25, pp. 28~30.

Southwell, R. and Gough, F., 1921, "The Free Transverse Vibration of Airscrew Blades," *British A.R.C. Reports and Memoranda*, No. 766.

DOT User Manual, Vanderplaates Research and Development, Inc., 1995.

Yntema, R., 1955, "Simplified Procedures and Charts for the Rapid Estimation of Bending Frequencies of Rotating Beams," NACA 3459.

Yoo, H., Ryan, R. and Scott, R., 1995, "Dynamics of Flexible Beams Undergoing Overall Motions," *J. of Sound and Vibration*, Vol. 181, No. 2, pp. 261~278.

Yoo, H. and Shin, S., 1998, "Vibration Analysis of Rotating Cantilever Beams," *J. of Sound and Vibration*, Vol. 212, No. 5, pp. 807~828.

Yun, Y. and Yoo, H., 2001, "Shape Optimization of Rotating Cantilever Beams Considering Modal and Stress Characteristics," *Transactions of the KSME*, A, Vol. 25, No. 4, pp. 645~653.